TECHNICAL NOTES

Elastic formulae for lateral displacement and rotation of arbitrarily-shaped embedded foundations

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INTRODUCTION

Foundations are frequently designed to transmit horizontal forces and/or moments. Such loading arises, for example, from wind, water and earth pressure, as well as from dynamic loads due to earthquakes, man-induced vibrations. and impacts. The list of special structures that impose significant lateral forces and moments includes transmission towers, radar antenna structures, large-span frames, and structures with heavy eccentric loads. Predicting and limiting the resulting horizontal displacement and rotation is a key geotechnical consideration, especially with many modern deformation-sensitive structures. Assessing the effects of soil-structure interaction on the behaviour of a superstructure involves estimating the ground compliance to such imposed lateral and moment loading.

A review of the literature reveals that only a very small number of solutions to this problem are presently available; several of them are summarized in tabular and graphical form in Poulos & Davis (1974). Most of these solutions refer either to uniformly distributed shear tractions or to linearly varying normal pressures applied directly to the surface of a homogeneous halfspace, and approximately simulating the action of a foundation carrying a horizontal force or a bending moment (Giroud 1968, 1969, 1970; Gerrard & Harrison 1970a and b). A few results are available for the more realistic case of loading applied through a rigid mat (Borowicka, 1943; Bycroft, 1956; Muskelishvili, 1963; Barkan, 1962; Gerrard & Harrison, 1970a and b). Solutions for embedded foundations are mainly available in the soil dynamics literature; they are recovered from the respective harmonic compliances when the frequency of oscillation approaches zero (Gazetas, 1983, 1987). Some static solutions have also been published for laterally loaded embedded foundations (e.g. Haritos & Keer, 1980; Johnson, Christiano & Epstein, 1975). Most of these solutions assume perfect contact, over the full depth of embedment, between the vertical foundation sidewalls and the surrounding soil.

This Note presents simple algebraic expressions (Table 1) for estimating the elastic horizontal displacement and rotation of arbitrarily-shaped rigid foundations partially or fully embedded in a reasonably uniform and homogeneous half-space. The shortcomings of modelling the soil behaviour as linearly elastic and the soil deposit as a homogeneous half-space are well understood and need no further elaboration. However, by choosing these models, the number of independent problem parameters is kept to a minimum, while trying to quantify the role of partial embedment and basemat shape. Of course, key to the success of the developed formulae in practical applications is the selection of an appropriate value for the secant Young's modulus. Such a selection can be based on experience, guided by theoretical results for idealized foundation shapes on non-linear and inhomogeneous soil deposits.

The development of the proposed expressions is based on an improved qualitative understanding of the role of embedment and of foundation shape, substantiated quantitatively bv the numerical results of an extensive rigorous parametric study using a Boundary-Element formulation. Table 2 lists the cases studied; they encompass a variety of basemat shapes, relative depths of embedment (D/B), values of Poisson's ratio (v), and relative heights of effective sidewallsoil contact (d/D). The meaning of the symbols is shown in Table 1. It is emphasized that the proposed formulae are essentially curve-fits to the theoretical data; hence the accuracy of the outcome may not be better than 10 or 20%. However, discrepancies of about 10% and sometimes much more are also observed among rigorous solutions; they arise due to different assumptions regarding the behaviour of soilfoundation interface ('rough' versus 'smooth' contact) different solution methods (integral transform techniques, semi-analytical procedures, finite-element and boundary-element formulations) and different degrees of precision with which the calculations are carried out.

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Loading		Surface foundation	Foundation p in open tree	aced ch
		$ \begin{array}{c} 1 \\ P_{i} \\ P_$		<u>+</u> □-+
	Lateral direction y	Longitudinal direction x	Both direct	ons
Horizontal	$\delta_0 = \frac{P}{EI}($	$2 - v(1 + v)\mu_0$	$\delta_{\rm trench} \simeq \delta_0 \mu_{\rm trench}$	(2)
	$\mu_{0_{y}} \simeq 0.5 - 0.28 \left[\frac{A_{b}}{4L^{2}} \right]^{0.45} $ (1a)	$\mu_{0_{s}} \simeq \left[1 + 1 \cdot 12 \left(\frac{1 - B/L}{1 - v}\right)^{1 \cdot 6} \right] \mu_{0_{s}} $ (1)	b) $\mu_{\text{trench}} \simeq 1 - 0.14 \left(\frac{D}{B}\right)^{-1}$. ³⁵ (2a)
Moment	$\theta_{0_{y}} = \frac{M}{E} \frac{2(1-v^{2})}{I_{y}^{0.75}} \mu_{0_{y}} $ (4)	$\theta_{0_x} = \frac{M}{E} \frac{2(1-v^2)}{I_x^{0-75}} \left(\frac{B}{L}\right)^{0.25} \mu_{0_x}$	$\theta_{aaa} \simeq \theta_{aaa}$	(9)
	$\mu_{0,\gamma} \simeq 0.43 - 0.10 \frac{B}{L} \tag{4a}$	$\mu_{0,x} \simeq 0.33 (B/L)^{0.15} \tag{2}$	a) (a)	
Loading	Embed	ded foundation with sidewall-soil contact surface of	otal area A "	
	Lateral direction y	Longitudinal d	rection x	
Horizontal		$\delta = \delta_{\rm french} \mu_{\rm wall}$		(3)
		$\mu_{\text{wall}} \simeq 1 - 0.35 \left[\frac{h}{B} \frac{A_{\text{w}}}{L^2} \right]^{0.2} \simeq \mu_{\text{wall}_{\text{x}}}$		(3a)
Moment		$\theta = heta_{\text{trench}} \mu_{\text{wall}}$		(2)
	$\mu_{\text{wall},,} \simeq \left\{ 1 + 0.92 \left(\frac{d}{L}\right)^{0.6} \left[1.5 + \left(\frac{d}{L}\right)^{1.9} \right] \right\}$	$\left(\frac{D}{d}\right)^{0.6} \right\}^{-1} \qquad (7a) \qquad \mu_{\text{wall},\text{a}} \simeq \left\{1 + 1.26 \frac{d}{B}\right[1]$	$+ \frac{d}{B} \left(\frac{D}{d} \right)^{0.2} \left(\frac{B}{L} \right)^{0.2} \right)^{-1}$	(1p)

Table 1. Proposed elastic formulae for estimating the horizontal displacement and rotation of arbitrarily shaped foundations

440

TECHNICAL NOTES

Table 2. List of the cases studied parametrically with boundary elements. Results of these studies formed the basis of the proposed formulae (Table 1) and are plotted as data points in Figs 2 and 4

Shape	L/B	D/B	d/B	ν
Square	1	0 1 2	$0 \\ 0, \frac{1}{2}, \frac{3}{4}, 1 \\ 0, \frac{1}{4}, \frac{2}{3}, \frac{3}{4}, 1$	0.25, 0.30, 0.33, 0.40, 0.45, 0.49 0.33, 0.40, 0.49 0.40, 0.40
Circle	1	0 0·5 1 2	0 1 1 1	all υ 0·33 0·33 0·33
Rectangle	1.5	0	0	0.25
	2	0 1 2	$0 \\ 0, \frac{1}{3}, 1 \\ 0, 1$	0·25, 0·30, 0·33, 0·40, 0·45 0·40 0·40
	3	0	0	0.25, 0.30, 0.33, 0.40
	4	0 0.5 1 1.5 2	$0 \\ 0, 1 \\ 0, \frac{1}{3}, \frac{2}{3}, 1 \\ 0, 1 \\ 0, 1$	0·30, 0·33, 0·40, 0·45 0·40 0·40 0·40 0·40 0·40
	5	0	0	0.25
	6	0 0·5 1 2	0 0, 1 0, 1 0, 1	0·40, 0·49 0·40 0·40, 0·49 0·40
	8	0	0	0.30
	10	0 0·5 1	$\begin{array}{c} 0\\ 0, 1\\ 0, \frac{1}{3}, \frac{2}{3}, 1 \end{array}$	0·33, 0·40 0·40 0·40
T-shape	1	0 1	0 1	0·40 0·40
	2	0 1	0 1	0·40 0·40
Strip	∞	$ \begin{array}{c} 0 \\ 0.5 \\ 1 \\ 1.5 \\ 2 \end{array} $	0 1 1 1 1	0·33, 0·50 0·25, 0·30, 0·33 0·25, 0·30, 0·33 0·25 0·33

This Note complements the study by Gazetas, Tassoulas, Dobry & O'Rourke (1985) for the vertical settlement of embedded foundations.

EXPRESSIONS FOR SURFACE FOUNDATION

For a foundation of area A_b and arbitrary (but solid) shape, one first determines a reasonable circumscribed rectangle 2B by 2L (L > B) as shown in Fig. 1 and Table 1. The results are *not* sensitive to the exact circumscribed rectangle.

The horizontal displacement δ_0 and rotation θ_0 produced by a horizontal force P_x (or P_y) and a moment M_x (or M_y) respectively, are given by equations (1) and (5) of Table 1 in terms of the Young's modulus and Poisson's ratio of the halfspace. In these expressions, I_x and I_y are the area moments of inertia of the actual basemat-soil contact surface around the x and y axes, respectively.

An idea of the scatter of the theoretical data points about the proposed expressions can be obtained from Fig. 2. Plotted in this figure are the dimensionless coefficients $\mu_{0_{rx}}$ and $\mu_{0_{ry}}$ for rocking around the x and y axes as functions of the aspect ratio B/L of the circumscribed rectangle. The



Fig. 1. Problem geometry: horizontal force (left) and moment (right) loading

scatter is substantially less for the horizontal displacements. In fact, the scatter about all other expressions given in Table 1 is considerably less than that shown in Fig. 2.

EFFECTS OF EMBEDMENT

There are three possible effects of embedment on the lateral displacement and rotation of a rigid foundation with vertical sidewalls. First, in reality, a foundation mat placed at a depth Dbelow the ground transmits loads to deeper soil layers which may be different (usually stiffer) than the layers affected by a similar surface foundation. Hence, other things being equal, increasing Dresults in different (usually smaller) foundation movement. However, when applying the proposed formulae in a practical situation, the engineer must first ensure that the deposit is indeed reasonably homogeneous, and then establish a representative value of soil modulus for the particular depth of embedment. (The deposit need not be very deep: horizontal and, especially, moment loading are known to produce very shallow 'pressure bulbs'.)

The other two effects of embedment here addressed are referred to as the 'trench' and the 'sidewall-contact' effects. They are illustrated in Fig. 3.

The 'trench' effect stems from the fact that even in a perfectly homogeneous half-space, the movement (δ_{trench} and θ_{trench}) of a foundation placed at the bottom of an open trench is smaller than that of the same foundation on the ground surface (δ_0 and θ_0). To understand why, one must visualize the horizontal plane passing through the base. For a surface foundation, this plane *is* the ground



Fig. 2. Development of the basemat shape factors μ_{rx} and μ_{ry} (for rotation about the x and y axes) in terms of the aspect ratio of the circumscribed rectangle



Fig. 3. 'Trench' and 'sidewall-contact' effects for horizontal and moment loading

surface, deforming free of any external stress. For an embedded foundation, normal and shear tractions from the overlying soil tend to restrict the deformation of this plane and thereby lead to smaller foundation movement. Careful examination of all the theoretical data reveals that the trench effect is negligible for moment loading; the effect of horizontal loading can be estimated from equation (2) of Table 1.

The sidewall-contact effect arises from the fact that, when the vertical sidewalls are in contact with the surrounding soil, part of the applied load is transmitted to the ground through normal and shear tractions from the sidewalls, thereby reducing the foundation movement. In addition to the analytical evidence developed in this work, there is ample experimental support of both the trench and sidewall-contact effects (e.g. Erden, 1974). Use of the analytical expressions given in Table 1 (equations (3) and (7)) will reveal that the sidewall effect is considerably more important than the trench effect for both of the loading cases considered.

In equation (3a) the symbol A_w denotes the total area of the sidewall-soil interface that is effective in transmitting load, while h is the depth



Fig. 4. Development of algebraic expressions of the factors for rotation about x (left) and translation parallel to x (right)

to the centroid of that interface. In order to indicate the accuracy of the proposed formulae, the dimensionless coefficients $\mu_{wall_{rx}}$ and $\mu_{wall_{x}}$ are plotted in Fig. 4 against two dimensionless expressions of geometric parameters. These latter expressions were derived by trial and error so that the numerical data plot within a very narrow band. Indeed, the scatter around the proposed curves is within 10%.

The expressions for $\mu_{wall_{rx}}$ and $\mu_{wall_{ry}}$, for rotation about x and y, having been developed in such a way as to minimize the overall scatter, do not become identical for L/B = 1, but differ by a negligible amount.

CONCLUSION

Simple algebraic formulae are presented (Table 1) for estimating the elastic horizontal displacement or rotation of a foundation subjected to a lateral force or moment, respectively. The formulae are valid only for a constant depth of embedment and for a solid basemat shape (rings and other annular shapes are excluded). However, the basemat may be of practically any shape, while the vertical sidewalls may have any degree of contact with the surrounding soil—from complete contact over the whole depth D to no contact at all.

The numerical data on which the proposed formulae are based were derived for an elastic and homogeneous half-space. In applications where realistic profiles are encountered, these formulae would still serve to obtain useful reference values and help to properly interpret the results of sophisticated (e.g. non-linear inelastic) analyses. Although most of the numerical data are for perfectly rigid basemat and walls, the proposed formulae would yield sufficiently accurate estimates of the average displacement and rotation of realistic flexible foundations.

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